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Symmetry in catastrophe theory, phase diagrams and two-dimensional conformal field theories*

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Abstract. The classification of symmetric catastrophes is studied to obtain Landau potentials for statistical models. Potentials for symmetric models with two order parameters are thoroughly discussed. The double-cusp catastrophe is used for illustration. Its various symmetries and corresponding statistical models are revealed. As an important example, the three-state Potts model is studied in detail. Emphasis is placed on connections with exact results from conformal field theories describing 2D symmetric models.

1. Introduction

The study of phase transitions with several order parameters has mainly been motivated by the crystallographic symmetry-changing transitions of solid state physics [1]. These symmetries are composed of discrete translations and rotations. The latter constitute the point groups, namely the finite subgroups of O(3). Their representations on the order parameters are well known [1, 2]. Alternatively, one may regard groups acting on internal variables of statistical models (electric or magnetic systems, etc) or field theories. The appearance of critical behaviour is frequently associated with a change in the internal symmetry of a system.

Catastrophe theory (also known as singularity theory) has been recognized as a rigorous mathematical framework to deal with the topological features of phase diagrams [3]. The catastrophe description, because of its invariance under diffeomorphisms, is particularly suited for studying internal symmetry in an abstract way, without relying on a privileged set of variables. For the same reason, the standard catastrophe theory seems to hide the symmetry-breaking character of some transitions, because it is difficult to tell by inspection if a given catastrophe has some symmetry and which symmetry it is.

The role of symmetry in catastrophe theory has already attracted some attention, mainly from the mathematical point of view [4-6], as a restriction on the allowed diffeomorphisms. Here stems the concept of symmetric catastrophes, which exhibit the symmetry explicitly and can be classified according to it. This classification suffices to solve the problem of symmetry in general, given the fact that all possible symmetries of catastrophes can be determined, as we will see.

The connection of catastrophe theory with Landau theory is mutually beneficial, because it allows transfer of information on catastrophe unfoldings and bifurcation

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sets to phase diagrams of statistical models and vice versa. As an example, Dynkin diagrams of catastrophes [7] describe the topology of extremal-point structures [6, 8], encoding the structure of phase diagrams. The way a Dynkin diagram unfolds (its so-called bloom) [6, 8] represents how new minima (phases) appear, conveniently exhibiting the phase structure. Moreover, Dynkin diagrams strongly suggest the discrete statistical models to be associated with them and their symmetry. In fact, lattice models with a state variable taking values on a Dynkin diagram have already appeared in the literature for the 2D case [9, also 10].

In this paper we shall focus upon the problems mentioned above; namely, classifying the possible symmetry groups for Landau potentials and studying the effect of symmetric as well as symmetry-breaking fields, hence revealing the phase diagram structure. We restrict to two order parameters (co-rank 2) for a thorough study, for mathematical reasons. However, the main features and possibilities of the approach will be well represented. The case of the permutation groups, which are realized by the Potts models, would demand higher co-rank and only a brief comment is made. For the description of the unfolding, we shall take as example the double-cusp catastrophe, which is the simplest compact case (a potential is called compact when the domain for which it is less than any given value is compact). It also exhibits the phenomenon of modulus; namely, an extra parameter that does not change the topology of the phase diagram (first considered in [11]). Its phase diagram includes the potentials for the *XY*, Ashkin–Teller and three-state Potts models.

Although we do not consider any space dimension in particular (the physical case being three), our classification is applied in full only in the 2D case, where arbitrarily high-order potentials are allowed by dimensional requirements (the elementary field is dimensionless). Besides, in that dimension the diverse types of critical behaviour are classified and completely described by 2D conformal field theory (2dCFT) methods [12], the external discrete symmetry being an important factor. Therefore, it is an appropriate situation to test how useful the symmetric catastrophe description can be. Actually, the observation of the coincidence of the ADE classification of catastrophes [5] and that of 2dCFT [13] has currently caused a revival of it [see also 14].

Before proceeding, it may be convenient to assess the adequacy of catastrophe theory in the classification of phase transitions. The assumption of analyticity at the critical point makes it equivalent to Landau theory, which is known not to hold in general in the scaling region. Nevertheless, the topological features of the phase diagram are not affected by this non-analyticity and they can always be represented by a Landau potential with the appropriate dependence of its coefficients on the thermodynamical parameters.

The paper is divided in three parts: first, we expose the methods to find group invariants and we describe symmetric catastrophe germs and their unfolding. We relate them to the generic catastrophes. Second, we study the double-cusp catastrophe and its phase diagram in the D_3 and D_4 symmetry cases. Third, we apply the previous results to the 2D models using some results of 2dCFT to establish the potentials for special series of models.

2. Group theory and catastrophe theory

Let us briefly review some concepts in catastrophe theory and in invariant theory before constructing the symmetric catastrophes. Catastrophe germs are classified, in first instance, by the order of the first non-null jet (truncated Taylor expansion). Let it be k. Then, for homogeneous cases, there exist local coordinates for which it is a k-degree monomial, written for co-rank 2 as

$$j^k(f)=\sum_{i+j=k} c_{ij}x^iy^j. \tag{2.1}$$

The action of diffeomorphisms on it reduces to linear transformations, GL(2, $\boldsymbol{R}$). Therefore, the second step consists of an algebraic classification of k-forms under the linear group. This is done by analysing the root structures of the equation $j^k(f)=0$ [15]. Since we have just two variables, after dehomogenization by factoring x^k out

$$j^k(f)=x^k\sum_j c_{ij}(y/x)^j=x^k\sum_j a_jt^j \qquad i=j-k \qquad t:=y/x \qquad a_j:=c_{ij} \tag{2.2}$$

we get an ordinary algebraic equation of degree k

$$\sum_j a_jt^j=0. \tag{2.3}$$

The root structure is given by the multiplicity of the roots of this equation and the number of real ones. It is invariant under linear transformations of the x, y variables that induce projective transformations on t. They form a group called PSL(2, $\boldsymbol{R}$). Each root structure corresponds to a class of germs giving the same catastrophe, which is represented by a canonical form. The compact germs have no real roots.

If the germ (2.1) is invariant under some discrete group, it must leave invariant not only the root structure but also the position of the roots; that is, it induces permutations among them. Thus, the symmetry groups of these germs are the finite subgroups of PSL(2, $\boldsymbol{R}$). These are well known to be isomorphic to the finite subgroups of O(2) [16], namely, the two series of cyclic and dihedral groups, C_n and D_n. Finding the symmetry of a given germ is not easy. It is necessary to find which projective transformations permute the roots. This is not yet the solution, because it only ensures the semi-invariance of the germ (invariance up to a constant, see [17] for definitions). The actual symmetry group is thus a subgroup of those transformations. The inverse process is equivalent but more fruitful: since we know all possible groups of symmetry, begin with the 2D representations of the groups and find all their invariants (as done in [17]). The canonical germs must be constructed from them and will exhibit the symmetry explicitly.

Complex germs [5] also have physical interest [13, 24]. For them, the groups of symmetry are the finite subgroups of PSL(2, $\boldsymbol{C}$), which are the C_n and D_n series and the three exceptional groups of symmetry of platonic polyhedra, T, O and I. The 2D representations and their invariants are built in [17]. They form a ring with a finite basis. With similar methods, it is easy to build the invariants of PSL(2, $\boldsymbol{R}$): starting from the complex representations of C_n by the roots of unity on two independent complex variables, z and $\bar{z}$, we can obtain the real ones in the standard way, just taking the real and imaginary parts of one of the two variables, $z=x+\mathrm{i}y$ (the other gives the complex conjugate representation $z^*=x-\mathrm{i}y$, see [18]). The basis has three complex invariants, $z\bar{z}$, z^n and $\bar{z}^n$, which after restriction to $\bar{z}=z^*$ lead to: $I_0:=zz^*=x^2+y^2$, which is invariant under the full O(2), and the real combinations $I_1:=z^n+z^{*n}$ and $I_2:=\mathrm{i}(z^n-z^{*n})$. To enlarge C_n to D_n we add the reflection $y\to -y$ (equivalent to $z\leftrightarrow z^*$). Observe that this representation of the reflection is real, and therefore different from the complex representation used in [17]. It leaves invariant the first but not the second and we have a basis with only two monomials, I_0 and I_1. They generate the ring of invariants, which is written $\boldsymbol{R}[I_0, I_1]$. In the case of C_n the three invariants must satisfy a relation (called syzygy) which turns out to be $4I_0^n + I_1^2 + I_2^2 = 0$, and the ring of invariants is the direct sum $\boldsymbol{R}[I_0, I_1] \oplus I_2 \boldsymbol{R}[I_0, I_1]$.

Any symmetrical homogeneous germ can be constructed taking a linear combination of powers of the basic monomials that fits the desired degree for it. The simplest case occurs for degree n. At this point, it is necessary to recall that only compact germs are relevant for potentials, because they are bounded below. But the nth order invariants I_1 and I_2 have only real roots, as is easily checked, implying they are non-compact. This can be solved for n even by adding to them $2a(x^2+y^2)^{n/2}$ (with a suitable $a \in \boldsymbol{R}$), whereas for n odd (2.3) has at least one real solution. As a last remark, all linear combinations of I_1 and I_2 have D_n symmetry because there is always a reflection axis, although it may not be distinguished unlike the x-axis selected before. With these provisos, we have the nth order invariant (n even), expressed in the complex form as

$$D_n: \quad z^n + z^{*n} + 2a(zz^*)^{n/2}. \tag{2.4}$$

The constant a is a symmetry-preserving modal parameter. As an aside, we notice that the total number of modal parameters for this germ is $n-3$ but the others break the symmetry. It can be seen that the real roots disappear for $a > 1$. The border cases $a = 1$ correspond to degenerate germs, with $n/2$ couples of real roots, e.g. $(z^{n/2} + z^{*n/2})^2$.

We shall show examples for the cases $n = 4, 6$. Formula (2.4) gives for $n = 4$, using real variables x, y,

$$D_4: \quad x^4 + y^4 - 6x^2y^2 + a(x^2+y^2)^2 = (a+1)\left\{x^4 + y^4 - 2\frac{3-a}{1+a}x^2y^2\right\} \tag{2.5}$$

equivalent to the usual form $x^4 + y^4 + \alpha x^2y^2$, for $-2 < \alpha < 2$. These two border values correspond to the two possible degenerate germs $(x^2-y^2)^2$ and $(x^2+y^2)^2$, only the second being compact. For $n = 6$

$$\begin{aligned} D_6: \quad & x^6 - y^6 - 15x^4y^2 + 15x^2y^4 + a(x^2+y^2)^3 \\ & = (a+1)x^6 + (a-1)y^6 - (15-3a)x^4y^2 + (15+3a)x^2y^4 \end{aligned} \tag{2.6}$$

where the border germs are $(x^3 - 3xy^2)^2$ and $(x^2+y^2)^3$. Writing the germ in the usual canonical form with the three modal parameters

$$x^6 + y^6 + \alpha x^4y^2 + \alpha' x^2y^4 + \beta x^3y^3 \tag{2.7}$$

we have that in the general case only the C_2 inversion symmetry is present, whereas for $\alpha = \alpha'$ there also exists the $x \leftrightarrow y$ symmetry, increasing to D_2. If besides $\beta = 0$, it enlarges up to D_4, as we realize by observing that the expression

$$x^6 + y^6 + (a+1)x^4y^2 + (a+1)x^2y^4 = (x^4 + y^4 + ax^2y^2)(x^2+y^2) \tag{2.8}$$

inherits the symmetry of the first factor. The D_6 case (2.6) is given by a uniparametric family of α, α' values. The generalization to larger n may be made along the same lines. It is interesting to note that all the groups D_k with $k \leq n$ and even, are realized in the moduli space of the n-degree germ.

It must be noticed that some invariants may accidentally have higher symmetry. They correspond to degenerate catastrophe germs: most germs have moduli; certain points on the boundary of the moduli space touch other moduli spaces of germs with higher symmetry, a phenomenon well known in algebraic geometry. Several examples have appeared above. A special one is the most general C_n invariant of nth degree that always has full D_n symmetry, as already mentioned.

For three or more variables the classification of invariants is not complete [17]. However, for groups of physical interest, like the symmetric group, it is still possible to establish their invariants [5]. The natural representation of S_N in $\boldsymbol{R}^N$ can be reduced to $\boldsymbol{R}^{N-1}$ making null the linear invariant. The symmetry is then realized as that of the N-simplex. The $N-1$ remaining invariants form a basis and a catastrophe germ may be built by any combination of them. The simplest case S_3 is already included in the previous classification, since $S_3 \approx D_3$. The S_3 representation was implicitly used in the three-state model of [21]. It is easy to check that the invariants are the same.

3. Symmetry of the unfolding: symmetry-restricted catastrophes

Once the symmetric germs have been classified, the first thing one observes is that arbitrary perturbations (deformations in the mathematical jargon) are likely to break the symmetry. Therefore, it is necessary to look for the symmetry-preserving perturbations. Alternatively, it is possible to construct catastrophe theory in a symmetric way from the beginning [5, 6], restricting to equivariant diffeomorphisms (symmetry-preserving diffeomorphisms, that is to say commuting with the action of the group) [4]. There is an equivariant Morse lemma [5]; the other elements of catastrophe theory can be specialized in the same way. This approach has been followed in a somewhat different context in [21].

For every symmetric function one can construct a symmetry-restricted Taylor expansion from the basis of invariants. As in the general case, the control parameters allow us to kill off some of the first Taylor coefficients, but the infinite tail still remains. It is removed by an equivariant transformation, leaving a term, the symmetric germ. This construction is the mathematical foundation for the work on symmetry of Landau potentials already cited [1, 2]. For instance, the most general D_n invariant potential is an arbitrary function of the two primitive invariants, I_0 and I_1, which were obtained in the previous section. The potential can thus be written as

$$\Gamma(I_0, I_1) = c_{10}I_0 + c_{01}I_1 + c_{20}I_0^2 + c_{02}I_1^2 + c_{11}I_0I_1 + \dots \tag{3.1}$$

The highest singularity arises for the value of the control parameters such that the maximum number of lowest order c_{ij} become null. These c_{ij} can be taken as new control parameters. The canonical germ arises after all possible higher-degree terms are eliminated by a nonlinear equivariant transformation.

A nonlinear equivariant transformation must transform invariants into invariants, producing just a change of variables in the function Γ. Therefore it might seem that the canonical forms for those functions should be the same as those for the generic co-rank 2 catastrophes. Such a fortunate fact is by no means true, because the converse property does not hold, namely, not every change of two variables can be produced as a transformation of the two invariants. This happens in spite of the fact that those changes of two variables that actually occur are parametrized by two arbitrary functions of I_0 and I_1.

When the control parameters are null we obtain the degenerate configuration with all equilibrium points coinciding. Non-null values (symmetric perturbations) may unfold them, producing spontaneous symmetry breaking. The first term is the O(2) quadratic invariant, that is, the Morse part of the potential, which always appears and is associated with the thermal perturbation, the coefficient being proportional to $t := T - T_c$. When $c_{10} < 0$, the equilibrium state (non-ordered phase) splits into n different minima, leading to ordered phases with spontaneously broken symmetry. Under the action of the C_n generating element

$$z \to e^{i(2\pi/n)} z \tag{3.2}$$

each phase transforms in the next one. They constitute the fundamental representation of the group. Considering the equivalence between phases and states of the statistical models, we deduce that these potentials are appropriate for the famous clock models [20].

The remaining unfolding terms in (3.1) lead to other patterns of spontaneous symmetry breaking. Their control parameters (coupling constants in field-theory language) may also depend on the temperature. However, we adopt the point of view of considering also non-symmetric perturbations, namely the entire unfolding of the catastrophe regarded as generic. Asymmetric perturbations may break the symmetry completely, e.g. the elementary fields (u, v) coupled to (x, y), or may still leave some subgroups unbroken. In this case, for non-null values of these control parameters, the irreducible representation of the group breaks into irreducible representations of the subgroup.

The best way to exhibit the phase structure as a function of the control parameters is by using Dynkin diagrams, which encode the relevant information on the extremal points of the potential [6]. The method starts from the catastrophe point and proceeds by splitting one minimum in each step, leaving a simpler catastrophe point. This reduces the codimension to the next lower multicritical-point set, represented by a subdiagram of the Dynkin diagram. Symmetric perturbations are easily controlled, because they yield symmetric Dynkin diagrams (a n-gon for the thermal perturbation $T < T_c$). Even for the simplest cases, constructing the entire phase diagram demands a highly developed spatial imagination. Some sections of the phase diagram of the double-cusp catastrophe will be analysed in the next section.

4. Symmetry in the double-cusp catastrophe and related statistical models

The X_9 catastrophe (2.5) is the simplest co-rank-2 catastrophe suitable for describing phase transitions [11]. Its modal parameter $a \equiv c_{22}$ labels a line of tetracritical points, divided in three parts, $a < -2$, $-2 < a < 2$, $a > 2$. The first one corresponds to non-compact germs. The other two parts correspond to compact germs but they are interchanged under a $\pi/4$ rotation on the (x, y) plane and one of them suffices.

The $a = 2$ value yields the O(2) degenerate catastrophe germ already mentioned. Therefore, it can be associated with the XY model. However, the symmetry can be restricted by third-degree terms. These are four independent monomials, but the two partial derivatives

$$\frac{\partial}{\partial x}(x^2 + y^2)^2 = 4x(x^2 + y^2) \tag{4.1}$$

$$\frac{\partial}{\partial y}(x^2 + y^2)^2 = 4y(x^2 + y^2) \tag{4.2}$$

cannot appear. The two remaining independent polynomials can be chosen to be the D_3 invariants $x(x^2 - 3y^2)$ and $y(y^2 - 3x^2)$. The third-degree perturbation is an arbitrary linear combination of them and also has D_3 symmetry, because it can be put in the canonical form x^3-3xy^2 by a rotation, that leaves the fourth-degree term invariant. Adding the thermal perturbation we get the potential

$$\Gamma=(x^2+y^2)^2+c(x^3-3xy^2)+w(x^2+y^2) \tag{4.3}$$

proposed some time ago for the three-state Potts model [23]. It has the form (3.1). The different Dynkin diagrams are shown in figure 1. The catastrophe point, $c=w=0$, is a symmetrical tricritical point.

The full unfolding includes four asymmetric terms, and can be written as

$$\Gamma=(x^2+y^2)^2+c(x^3-3xy^2)+w_1x^2+w_2y^2+2w_{12}xy+ux+vy. \tag{4.4}$$

The codimension is 6, as the seven possible extremal points (figure 1) indicate. The three-component model of [21, 22], with codimension 5, can be represented by this potential, after a change of variables (for it, $c=4/3$). It was studied with the renormalization group and shown to have a symmetric tricritical point (TCP) corresponding to the three-state Potts model. The phase diagram obtained there can be interpreted as a part of that of (4.4), as we shall see.

Let us introduce the change of parameters

$$w_+=(w_1+w_2)/2 \qquad w_-=(w_1-w_2)/2. \tag{4.5}$$

A simple phase diagram for (t, u, v) occurs when $w_{12}=w_-=0$, corresponding to the Potts model with external fields. If we assume that the only dependence on t resides in w, we obtain the phase diagram with three TCPs placed symmetrically at non-vanishing values of (u, v) and a quadruple point at $u=v=0$, which corresponds to a discontinuous phase transition. If c also vanishes when $t=0$, we get the symmetrical TCP [23]. Presumably, when w_{12}, $w_-\neq 0$, we can still manage to produce a TCP, obtaining the three tricritical lines arranged in a symmetric fashion of [22].

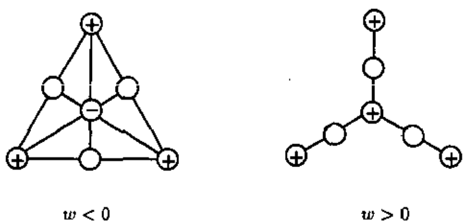

Figure 1. Dynkin diagrams for the D_3 unfolding of the double cusp.

The phase diagram for (w_1, w_2) when $u=v=w_{12}=0$ can be obtained analytically although the insight gained from the Dynkin diagrams is still fundamental to organize it. It illustrates the possible Dynkin diagrams associated with this catastrophe, and therefore the connection of the Potts model with others also described by (4.4). This diagram is exhibited in figure 2, for a non-zero value of c. Some conclusions about the whole phase diagram can also be drawn just with the help of symmetry considerations, but we will not dwell on it.

For $a>2$ (equivalently, $-2<a<2$) the double-cusp is more complicated. The partial unfolding

$$\Gamma=x^4+y^4+ax^2y^2+w_1x^2+w_2y^2 \tag{4.6}$$

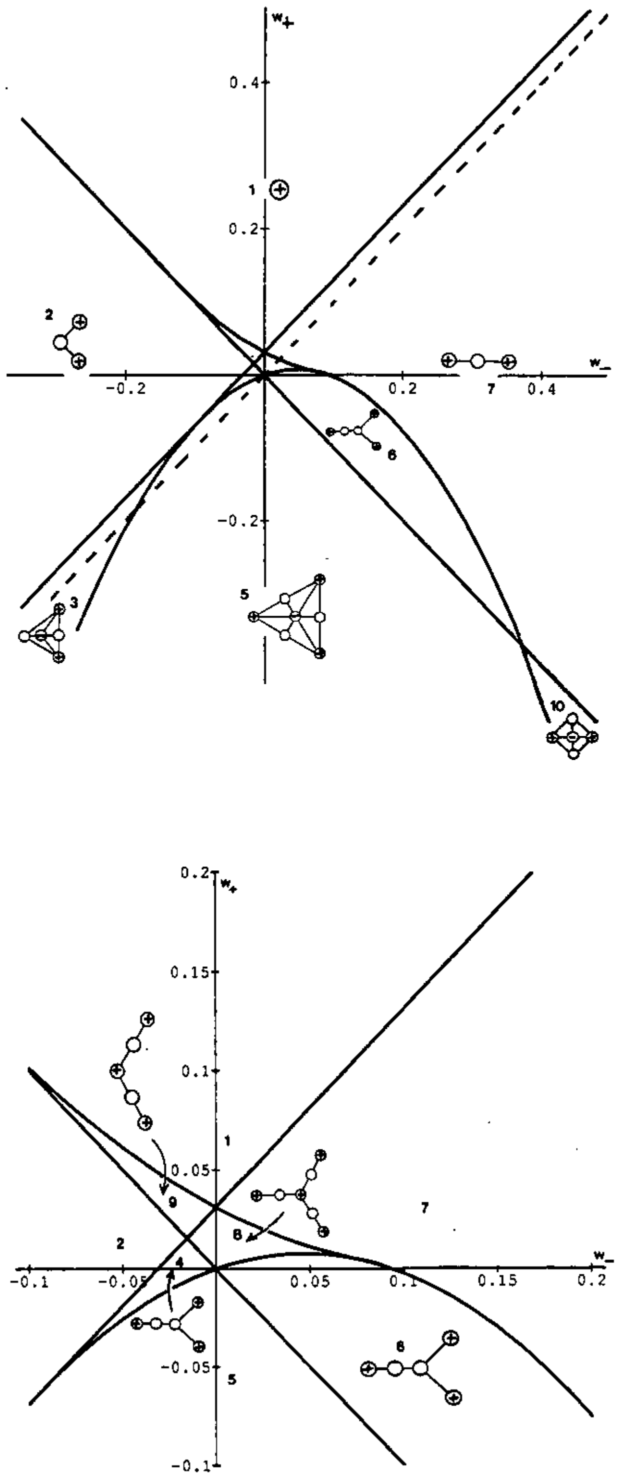

Figure 2. Phase diagram for $u = v = w_{12} = 0$ $(c = 1/3)$ showing characteristic Dynkin diagrams. The quadruple point is on the segment of the w_+ axis located in zone 8.

was analysed in [11]. When $w_1 = w_2 = w < 0$ a spontaneously broken D_4 configuration appears, with four minima, four saddle points and one central maximum forming a square, in place of the triangle of figure 1. It recalls the tensor product structure $X_9 = A_3 \otimes A_3$ plus a four-spin interaction (A_3 is the single cusp that corresponds to the Ising model). It can therefore be associated with the Ashkin–Teller model.

5. Series of symmetric potentials and 2D conformal field theories

As we saw in section 3, for every symmetry group there is a whole series of symmetry-restricted catastrophes. These series now have an increasing interest because of their connection with the problem of classification of the 2DCFTs [10, 13, 26], with regards to the ground state problem of string theory. A Zamolodchikov and V Fateev have described [25, 26] several series of models for some groups, namely, $\boldsymbol{Z}_2$ and $\boldsymbol{S}_3$ ($\boldsymbol{C}_2$ and $\boldsymbol{D}_3$ in our notation). There is one series for the first one (the minimal models with diagonal modular matrix) and two for the second one, the W_3-symmetric and the 4/3-parafermion series. The central charges are

$$c_1 = 1 - \frac{6}{p(p+1)} \tag{5.1}$$

$$c_2 = 2\left(1 - \frac{12}{p(p+1)}\right) \tag{5.2}$$

$$c_3 = 2\left(1 - \frac{12}{p(p+4)}\right) \tag{5.3}$$

whose limits when p goes to infinity are 1 and 2, indicating that the necessary number of order parameters are 1 and 2, respectively. The $\boldsymbol{Z}_2$-series potentials have the form $x^{2(p-1)}$ and correspond to the critical behaviour of 1D spin models [26, 27]. The potentials for the two-order-parameter series are not yet well established.

Zamolodchikov's procedure to associate a Lagrangian with a 2dCFT starts by finding in the algebra of fields a reduced set (elementary fields) capable of generating the others as composites. This gives the co-rank, while the codimension is given by the number of relevant fields. Taking into account the external symmetry, it should already be possible to determine the potential. Nevertheless, an independent check is available, coming from some operator product expansions (OPEs) which demand the presence of double derivatives of the elementary fields (descendant fields) and are consequently interpreted as a sort of equations of motion. The procedure works very neatly for the $\boldsymbol{Z}_2$-series, but for the others certain ambiguities may appear and, in some cases, different potentials have been found by different authors [28, 29].

An interesting case is the first member of the W_3 series, the critical three-state Potts model [30]. It has two elementary spin fields, σ and $\bar{\sigma}$, and seven relevant fields altogether, identifiable as

$$\Phi^{\pm}_{(33)} = \sigma \text{ and } \bar{\sigma}, \qquad \Phi^{\mp}_{(31)} = \sigma^2 \text{ and } \bar{\sigma}^2$$

$$\Phi_{(12)} \equiv \varepsilon = \sigma\bar{\sigma},\ \Phi_{(12,13)} = \sigma^3 \text{ and } \Phi_{(13,12)} = \bar{\sigma}^3.$$

This already suggests the potential (4.4) for it. In fact, it agrees with that found in [32] $(\sigma\bar{\sigma})^2 = (x^2+y^2)^2$, from the $\sigma\varepsilon$ OPE after identifying $\Phi^{+}_{(31)} = \bar{\sigma}^2$. Nevertheless, this field also appears in that OPE, and before the σ descendant, which implies that it must be included in the equation of motion. Therefore, the potential must include the cubic term, as it should be expected to convey the necessary D_3 symmetry. This reconciles it with the one in [28], as its natural compactification.

A similar analysis for the first member of the 4/3-parafermion series (the tricritical three-state Potts model) would suggest a potential such as

$$\Gamma=(x^2+y^2)^3+c(x^3-3xy^2)+g(x^2+y^2)^2+\ldots. \tag{5.4}$$

The cubic term is now irrelevant whereas the quartic is relevant (observe that this fact contradicts the naïve field dimensions). This model is connected to the previous one by the renormalization flow generated by the g perturbation.

One can guess the general form of the potential for both series, according to (3.1),

$$\Gamma(\sigma,\bar\sigma)=\sum c_{km}(\sigma\bar\sigma)^k(\sigma^3+\bar\sigma^3)^m. \tag{5.5}$$

An equivalent form was proposed in the first reference in [25], with $\sigma^{3m}+\bar\sigma^{3m}$ instead of $(\sigma^3+\bar\sigma^3)^m$, but it is less convenient because the germ must contain the latter (the former is invariant under the much larger group D_{3m}). But, as we have seen, power-counting arguments cannot be used and a detailed study of the OPEs for any particular model seems to be necessary. According to [29], the first term of the kth model of the W_3 series is

$$\Gamma(\sigma,\bar\sigma)=(\sigma\bar\sigma)^{k+1}. \tag{5.6}$$

However, more operators must appear to determine its symmetry and we are not able to find a general rule to assign a potential to every model.

Many more conformal models with discrete symmetries are suitable for a Lagrangian description. As was mentioned before, a good candidate for the potential of the Ashkin–Teller model is (4.6). Detailed studies of the 2D model seem to agree with it [10, 13, 32].

6. Summary and discussion

We have studied the role of symmetry in the context of catastrophe theory and how it helps to understand phase transitions in statistical models. From the invariance properties of catastrophe germs, we could determine the possible groups of symmetry for co-rank 2 (two order parameters), C_n and D_n. It is remarkable that this excludes many models from having a Landau potential description with two order parameters, for instance the q-state Potts model for $q\geqslant 4$. We saw next how to construct germs corresponding to those groups, which are at special points in their moduli space, and how to analyse the symmetry in the entire moduli space. A general perturbation of a germ breaks its symmetry, but we saw how the symmetry-restricted unfolding preserves the full symmetry. In this line, we applied symmetry considerations to analyse sections of the X_9 phase diagram and to relate them to known statistical models.

Finally, we attempted to match the symmetric potentials with 2DCFTs, known to have those symmetries. We have succeeded partially. A deeper analysis should use further information provided by the fusion rules. However, there may be a more illuminating and economical way, namely directly comparing a complete classification of symmetrical germs with a classification of 2dCFT with additional symmetry by modular invariance. Hopefully, there is an exact correspondence, as in the case of the ADE classification of simple germs and that of minimal models [13]; but in that more general case there do not seem to be complete classifications of either germs or modular invariant 2dCFT with which to begin.

One might wonder if other series of 2dCFT also admit a Landau potential description by our co-rank-2 catastrophes. This is a non-trivial question, because even if we have a construction of a theory in terms of a particular number of fields larger than two, it can still be possible to find another using just two. For instance, this happens for the Z_n parafermion models [31]. Actually, potentials with two fields for this series have been found recently [32], relying in a coset representation of it. They fit in (2.4).

As was observed in the introduction, the application of catastrophe theory reaches its full power in the 2D case, but it is in no way restricted to it. According to the Wilson ε-expansion, a model is defined in the dimension in which its renormalization group fixed point is trivial, namely, there are no anomalous dimensions and the renormalized Lagrangian is the exact effective potential. If we believe far enough in the ε-expansion, this potential also describes the phase transition in other dimensions, as long as it converges, although their critical points are non-trivial and the dimensions of fields anomalous. Somehow, the potential and the phase diagram that it entails can be considered independent of the space dimension, even though the dependence of its coefficients on the thermodynamical parameters may change with it, as we have seen for the three-state Potts model, leading to an apparent change of the phase diagram.

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